CONCISE DECISION DIAGRAMS FOR BRANCHING AND STORING NEAR-OPTIMAL SOLUTIONS

Thiago Serra       John Hooker

Carnegie Mellon University

MOPTA 2016
IF FINDING ONE IS EASY, WHAT ABOUT ALL?
Motivation

WE CARE ABOUT THAT

EVEN IN FICTION
Poké Ball  Belt  Bag
(Real) Motivation

ONE OPTIMAL SOLUTION MAY NOT BE ENOUGH
Alternative courses of action
Alternative courses of action

How much is left on the table?
Truthful bidding in combinatorial auctions
Truthful bidding in combinatorial auctions

Prices defined by solving problem many times
Solvers are already doing that
Solvers are already doing that

Where is my Poké Belt?
Our approach

USING AND RELAXING DECISION DIAGRAMS
From trees to diagrams

Too many solutions,
not much information

Variable for branching:

\[ \text{X1} \]

\[ \text{X2} \]

\[ \text{X3} \]

Branching tree
From trees to diagrams

Too many solutions, not much information

Variable for branching:

X1
X2
X3

Branching tree

Decision diagram

0-arc
1-arc
From trees to diagrams

Anything other than a path gets smaller

Branching tree
From trees to diagrams

Anything other than a path gets smaller

Branching tree

Decision diagram
From trees to diagrams

Anything other than a path gets smaller

Solutions $\equiv$ Leaves
From trees to diagrams

Anything other than a path gets smaller

Solutions \equiv \text{Leaves} \quad \rightarrow \quad \text{Solutions} \equiv \text{Paths}
From trees to diagrams

Anything other than a path gets smaller

Solutions $\equiv$ Leaves $\rightarrow$ Solutions $\equiv$ Paths

Optimal solutions $\equiv$ Shortest (Longest) paths
Sound relaxations

Exact for $\Delta$–optimal solutions

Weights of 1-arcs:

3

2

3

Decision diagram
Sound relaxations

Exact for $\Delta$–optimal solutions

Min problem: optimal value $z^*$ to $z^* + \Delta$

Weights of 1-arcs:

3

2

3

Decision diagram
$(z^* = 3 \quad \Delta = 3)$
Sound relaxations

Exact for $\Delta$–optimal solutions

Min problem: optimal value $z^*$ to $z^* + \Delta$

Weights of 1-arcs:

- 3
- 2
- 3

Decision diagram ($z^* = 3$, $\Delta = 3$)

Sound relaxation
Sound relaxations

Exact for $\Delta$–optimal solutions

Min problem: optimal value $z^*$ to $z^* + \Delta$

Weights of 1-arcs:

- 3
- 2
- 3

Decision diagram

($z^* = 3$, $\Delta = 3$)

Sound relaxation
Sound relaxations

Decision diagram ≡ Paths within a budget

Decision diagram

$z^* = 3 \quad \Delta = 3$

Sound relaxation
Sound relaxations

Decision diagram $\equiv$ Paths within a budget

Some arcs forbidden after worse choices

Decision diagram

Sound relaxation

$z^* = 3$  $\Delta = 3$
How it works

Reduction

Merges nodes with same children
How it works
Reduction
Merges nodes with same children
How it works

Reduction

Merges nodes with same children
How it works

Reduction

Merges nodes with same children

Converges to minimum size diagram
How it works

Sound-reduction

Redirects incoming arcs from node $u$ to $v$
How it works

Sound-reduction

Redirects incoming arcs from node u to v
How it works

Sound-reduction

Redirects incoming arcs from node $u$ to $v$

(i) Node $v$ has a superset of completions
How it works

Sound-reduction

Redirects incoming arcs from node $u$ to $v$

(i) Node $v$ has a superset of completions

(ii) Min cost $r-u$ path

+ Min cost extra $v-t$ path

$> z^* + \Delta$
How it works

Sound-reduction

\[ z^* = 1 \quad \Delta = 6 \]
How it works

Sound-reduction

$z^* = 1 \quad \Delta = 6$

Min $r-u_2$ path: 6

$+ \quad$ Min extra $q-t$ path: 2

$\geq 7$
How it works

Sound-reduction

\[ z^* = 1 \quad \Delta = 6 \]

Min r-u_2 path: 6

+ Min extra q-t path: 2

> 7
How it works
Sound-reduction

\[
\begin{align*}
z^* &= 1 \\
\Delta &= 6
\end{align*}
\]

Min r-u\textsubscript{2} path: \hspace{1cm} 6

+ Min extra v\textsubscript{2}-t path: \hspace{1cm} 2

\[
> 7
\]
How it works

Sound-reduction

\[ z^* = 1 \quad \Delta = 6 \]

Min r-u_2 path: 6

+ Min extra v_2-t path: 2

\[ > 7 \]
How it works

Sound-reduction

\[ z^* = 1 \quad \Delta = 6 \]

Min r-u\(_1\) path: 5

+ Min extra v\(_1\)-t path: 3

> 7
How it works

Sound-reduction

\[
z^* = 1 \quad \Delta = 6
\]

Min \ r-u_1 \ path: \ 5

+ Min extra \ v_1-t \ path: \ 3

\[
\text{total} = 8 > 7
\]
How it works

Sound-reduction

Same children → Extra paths are costly

\[ z^* = 1 \quad \Delta = 6 \]
How it works

Sound-reduction

Bottom-up comparison of completions

\[ u_3 \rightarrow \text{Min extra } v_3-t \text{ path: 2} \]
How it works

Sound-reduction

Bottom-up comparison of completions

\[ \text{Min extra } v_2 \text{-t path: 2} \]

\[ \text{Min extra } v_3 \text{-t path: 2} \]
How it works
Sound-reduction

Bottom-up comparison of completions

- $u_1 \rightarrow$ Min extra $v_1$-t path: 3
- $u_2 \rightarrow$ Min extra $v_2$-t path: 2
- $u_3 \rightarrow$ Min extra $v_3$-t path: 2
How well it works
Small problems
How well it works

Small problems

stein9

Size of Search Tree / Reduced Diagram (Semilog)

- Tree
- DD

stein15

Size of Search Tree / Reduced Diagram (Semilog)

- Tree
- DD
How well it works

Small problems
How well it works

Large problems

air01

Size of Search Tree / Reduced Diagram (Semilog)

# of Nodes

10^4

10^3

10^2

10^1

10^0

Δ

0 200 400 600 800 1,000

Tree

DD

Size of Reduced Diagram / Sound-Reduced Diagram

# of Nodes

10^5

10^4

10^3

10^2

10^1

0 200 400 600 800 1,000

DD

SndDD

lseeu

Size of Search Tree / Reduced Diagram (Semilog)

# of Nodes

10^4

10^3

10^2

10^1

10^0

Δ

0 50 100 150 200

Tree

DD

Size of Reduced Diagram / Sound-Reduced Diagram

# of Nodes

4

3

2

1

0 50 100 150 200

DD

SndDD
How well it works

Large problems
How well it works

Large problems

pipex

sentoy
What we learned

Sound relaxations disguise suboptimal solutions
What we learned
Sound relaxations disguise suboptimal solutions

If $\Delta=0$, there are no deviations
and no forbidden arcs
What we learned

Sound relaxations disguise suboptimal solutions

If $\Delta=0$, there are no deviations
and no forbidden arcs

When sound-reducing $u$ into $v$, $\min r-u$ path $> \min r-v$ path
What we learned

Sound-reduction converges to minimum diagram
What we learned

Sound-reduction converges to minimum diagram

Equivalent diagrams $\rightarrow$ Same completion sets
What we learned

Sound-reduction converges to minimum diagram

Equivalent diagrams $\rightarrow$ Same completion sets

$\{\phi\}$  $\{1\}$  $\{0,1\}$
What we learned

Sound-reduction converges to minimum diagram

Sound diagrams $\rightarrow$ Essential completion sets

\[
\begin{align*}
\{000, 010\} & \quad \{100, 101\} \\
\{00, 10\} & \quad \{00, 01\}
\end{align*}
\]
What we learned

What if the diagram is exact for $[z^*, z^* + \Delta]$?
What we learned

What if the diagram is exact for \([z^*, z^* + \Delta]\)?

Decision diagram
\((z^* = 1, \Delta = 0)\)

Generalized sound relaxation
What we learned

What if the diagram is exact for \([z^*, z^*+ \Delta]\)?

Minimizing the generalized relaxation

Solving the graph coloring problem

Decision diagram
\((z^* = 1 \quad \Delta = 0)\)

Generalized relaxation
What we learned

What if the diagram is exact for \([z^*, z^* + \Delta]\)?

Generalizing the relaxation makes it NP-hard

Minimizing the generalized relaxation

Solving the graph coloring problem

Decision diagram 
\((z^* = 1 \quad \Delta = 0)\)

Generalized sound relaxation
What we learned

What if we approximate or go heuristic?
What we learned

What if we approximate or go heuristic?
What we learned

What if we approximate or go heuristic?

Checking if two nodes can be unified

Solving a subset sum problem
What we learned

What if we approximate or go heuristic?

Every step may be co-NP-complete
What comes next

Branch less often
What comes next

Branch less often

We are still generating all tree nodes
What comes next

Branch less often

We are still generating all tree nodes
What comes next

Identify nodes with equivalent states
What comes next

Identify nodes with equivalent states

\[ x_1 + x_2 + \sum_{i=3}^{n} \alpha_i x_i \geq 3 \]
\[ x_1 + x_2 + \sum_{i=3}^{n} \beta_i x_i \geq 4 \]
What comes next

Identify nodes with equivalent states

\[
x_1 + x_2 + \sum_{i=3}^{n} \alpha_i x_i \geq 3
\]
\[
x_1 + x_2 + \sum_{i=3}^{n} \beta_i x_i \geq 4
\]

\[
x_1 = 0 \quad x_1 = 1
\]

\[
x_2 + \sum_{i=3}^{n} \alpha_i x_i \geq 3
\]
\[
x_2 + \sum_{i=3}^{n} \beta_i x_i \geq 4
\]

\[
x_2 + \sum_{i=3}^{n} \alpha_i x_i \geq 2
\]
\[
x_2 + \sum_{i=3}^{n} \beta_i x_i \geq 3
\]
What comes next

Identify nodes with equivalent states

\[
x_1 + x_2 + \sum_{i=3}^{n} \alpha_i x_i \geq 3
\]

\[
x_1 + x_2 + \sum_{i=3}^{n} \beta_i x_i \geq 4
\]
What comes next

Identify nodes with equivalent states

\[ x_1 + x_2 + \sum_{i=3}^{n} \alpha_i x_i \geq 3 \]
\[ x_1 + x_2 + \sum_{i=3}^{n} \beta_i x_i \geq 4 \]
What comes next

Identify nodes with equivalent states

Same RHS $\rightarrow$ Same completions
What comes next

Identify nodes with equivalent states

Same RHS $\rightarrow$ Same completions

Saturated constraints can be ignored
What comes next
Identify nodes with equivalent states

Resulting diagram may be a relaxation
What comes next
Identify nodes with equivalent states

Resulting diagram may be a relaxation

\[ z^* = 7, \Delta = 1 \]
What comes next

Identify nodes with equivalent states

Resulting diagram may be a relaxation

\[ z^* = 7, \Delta = 1 \]
What comes next

Identify nodes with equivalent states

Resulting diagram may be a relaxation

\[ z^* = 7, \Delta = 1 \]
What comes next
Identify nodes with equivalent states

Resulting diagram may be a relaxation

\[ z^* = 7, \Delta = 1 \]
How far we are
Small problems
How far we are

Large problems
Takeaway 1

DECISION DIAGRAMS
SAVE SPACE
Takeaway 2

RELAXED BRANCHING DOES LESS WORK
QUESTIONS?